Essential spectrum of local multi-trace boundary integral operators

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Abstract

Considering pure transmission scattering problems in piecewise constant media, we derive an exact analytic formula for the spectrum of the corresponding local multi-trace boundary integral operators in the case where the geometrical configuration does not involve any junction point and all wave numbers equal. We deduce from this the essential spectrum in the case where wave numbers vary. Numerical evidences of these theoretical results are also presented.

Introduction

Many applications involve the simulation of wave propagation phenomena in media with piecewise constant material characteristics that can be modeled by elliptic partial differential equations with piecewise constant coefficients. In such situations, the computational domain is naturally partitioned into sub-domains corresponding to the values of material characteristics.

As regards numerical approaches to be used to tackle wave propagation problems, although one may opt for finite elements or similar volume methods, boundary integral equation methods provide accurate alternatives that are less prone to such undesirable effects as numerical dispersion. Admittedly discretization of boundary integral equations leads to dense ill conditioned matrices which raises numerical challenges and requires careful implementation, but many progresses achieved in the past decade (efficient preconditioners, fast multipole methods, adapted quadrature techniques) now place boundary integral equation techniques as a serious numerical alternative for high performance computations.

In the context of parallel computing, it becomes desirable to embed integral equation methods into a domain decomposition paradigm. The Boundary Element Tearing and Interconnecting method (BETI) was developed in this spirit, more than a decade ago, as an integral equation counterpart of the FETI method, see [8, 13, 14, 17, 23]. An alternative approach dubbed Multi-Trace formalism, leading to different solvers, was introduced a few years ago [18, 19, 10, 11, 2, 3, 4, 5], providing other boundary integral formulations adapted to multi-domain geometrical configurations. Multi-trace boundary integral formulations seem well adapted to block preconditioners for domain decomposition but still very little is known about associated iterative global solvers. To our knowledge, the only contributions in this direction are [11, 7].

A precise knowledge of the spectral structure of the equation under consideration is most of the time a key ingredient for devising efficient domain decomposition strategies. The

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main purpose of the present contribution is thus to provide results concerning the spectrum of local multi-trace operators. In particular we describe the spectrum through an explicit formula in the case where the propagation medium admits uniform characteristic material parameters, which yields the location of the accumulation points of the spectrum of local multi-trace operators in the general case of piecewise varying effective wave numbers. In addition, the present contribution provides a proof of the well posedness of the local multi-trace formulations in the case of non-trivial relaxation parameters, which was mentioned in [11, Rem.2] as an open problem.

The present contribution is organized as follows. In the first section we describe the scattering problem and the geometrical configurations under consideration. In Section 2 and 3 we introduce notations related to traces and integral operators, and recall well established results of classical potential theory. In Section 4 we recall the derivation of (relaxed) local multi-trace formulations as presented in [11], and prove uniqueness of the solutions to this formulation for general values of the relaxation parameter. Section 5 presents detailed calculus achieved in the case of two particular elementary geometrical configurations. Section 6 is dedicated to the study of the spectrum of the local multi-trace operator, and Section 7 will present numerical results confirming the theory.

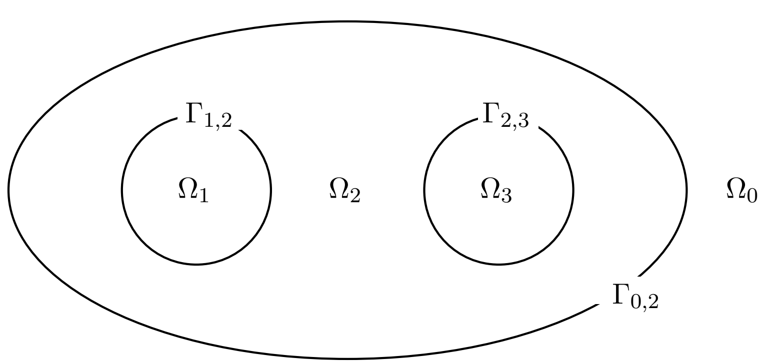
1 Setting of the problem

In this section, we will mainly introduce notations, and write the problem under consideration, starting from a precise description of the geometrical configurations we wish to examine. First of all, we consider a partition $\mathbb{R}^d := \cup_{j=0}^{n}\overline{\Omega}_j$ where the Ω_j's are Lipschitz domain, where $\mathbb{R}^d = \mathbb{R}, \mathbb{R}^2$ or $\mathbb{R}^3$. We assume that each Ω_j is bounded except Ω_0. Changing the numbering of sub-domains if necessary, we may assume without loss of generality that each Ω_j is connected. We shall refer to the boundary of each sub-domain by $\Gamma_j := \partial\Omega_j$, and also set $\Gamma_{j,k} := \Gamma_j \cap \Gamma_k = \partial\Omega_j \cap \partial\Omega_k$ to refer to interfaces. The union of all interfaces will be denoted

$$\Sigma := \bigcup_{j=0}^{n} \Gamma_j = \bigcup_{0\leq j<k\leq n} \Gamma_{j,k} \quad .$$

This set will be referred to as the skeleton of the partition. We make a further strong geometric hypothesis, assuming that the sub-domain partition under consideration does not involve any junction point, so that each $\Gamma_{j,k}$ is a closed Lipschitz manifold without boundary,

$$\partial\Gamma_{j,k} = \emptyset \quad \forall j,k = 0\ldots n. \tag{1}$$



Sobolev spaces We need to introduce a few usual notations related to standard Sobolev spaces. If $\omega \subset \mathbb{R}^d$ is any Lipschitz domain, we set $\mathrm{H}^1(\omega) := \{v \in \mathrm{L}^2(\omega),\ \nabla v \in \mathrm{L}^2(\omega)\}$ equipped with the norm $\|v\|^2_{\mathrm{H}^1(\omega)} := \|v\|^2_{\mathrm{L}^2(\omega)} + \|\nabla v\|^2_{\mathrm{L}^2(\omega)}$, and $\mathrm{H}^1(\Delta, \omega) := \{v \in \mathrm{H}^1(\omega),\ \Delta v \in \mathrm{L}^2(\omega)\}$ equipped with $\|v\|^2_{\mathrm{H}^1(\Delta,\omega)} := \|v\|^2_{\mathrm{H}^1(\omega)} + \|\Delta v\|^2_{\mathrm{L}^2(\omega)}$. In addition, if $\mathrm{H}(\omega)$ refers to any of the above mentioned spaces, then $\mathrm{H}_{\mathrm{loc}}(\overline{\omega})$ will refer to the space of functions v such that $\psi v \in \mathrm{H}(\omega)$ for any $\psi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}^d) := \{\varphi \in \mathscr{C}^\infty(\mathbb{R}^d),\ \mathrm{supp}(\varphi) \text{ bounded}\}$.

For any Lipschitz open set $\omega \subset \mathbb{R}^d$, we shall refer to the space of Dirichlet traces $\mathrm{H}^{1/2}(\partial\omega) := \{v|_{\partial\omega},\ v \in \mathrm{H}^1(\omega)\}$ equipped with the norm $\|v\|_{\mathrm{H}^{1/2}(\partial\omega)} := \min\{\|u\|_{\mathrm{H}^1(\omega)},\ u|_{\partial\omega} = v\}$. The space of Neumann traces $\mathrm{H}^{-1/2}(\partial\omega)$ will be defined as the dual to $\mathrm{H}^{1/2}(\partial\omega)$ equipped with the corresponding canonical dual norm $\|p\|_{\mathrm{H}^{-1/2}(\partial\omega)} := \sup_{v \in \mathrm{H}^{1/2}(\partial\omega)\setminus\{0\}} |\langle p, v\rangle|/\|v\|_{\mathrm{H}^{1/2}(\partial\omega)}$.

Transmission problem We will consider a very standard wave scattering problem (so-called transmission problem), imposing Hemholtz equation in each sub-domain, as well as transmission conditions across interfaces: find $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\left\{\begin{aligned}
&-\Delta u - \kappa_j^2 u = 0 \quad \text{in } \Omega_j \quad \forall j = 0 \dots n\\
&u - u_{\mathrm{inc}} \text{ is outgoing in } \Omega_0\\
\\
&u|_{\Gamma_j} - u|_{\Gamma_k} = 0\\
&\partial_{n_j} u|_{\Gamma_j} + \partial_{n_k} u|_{\Gamma_k} = 0 \quad \text{on } \Gamma_{j,k} = \Gamma_j \cap \Gamma_k,\ \ \forall j,k = 0 \dots n
\end{aligned}\right. \tag{2}$$

In the equation above $u_{\mathrm{inc}} \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a known source term of the problem satisfying $-\Delta u_{\mathrm{inc}} - \kappa_0^2 u_{\mathrm{inc}} = 0$ in $\mathbb{R}^d$. In addition, we assume that $\kappa_j > 0$ for all $j = 0 \dots n$. Problem (2) is known to admit a unique solution, see [20] for example. The outgoing condition in (2) refers to Sommerfeld's radiation condition, see [15]. A function $v \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \Omega_0)$ will be said κ-outgoing radiating if $\lim_{r\to\infty} \int_{\partial \mathrm{B}_r} |\partial_r v - \imath\kappa v|^2 d\sigma = 0$ where B_r refers to the ball of radius r and center $\mathbf{0}$, and ∂_r is the partialderivative with respect to the radial variable $r = |\boldsymbol{x}|$.

Trace operators As this problem involves transmission conditions, and since we are interested in boundary integral formulations of it, we need to introduce suitable trace operators. According to [22, Thm. 2.6.8 and Thm 2.7.7], every sub-domain Ω_j gives rise to continuous boundary trace operators $\gamma^j_{\mathrm{D}} : \mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}_j) \to \mathrm{H}^{1/2}(\partial\Omega_j)$ and $\gamma^j_{\mathrm{N}} : \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j) \to \mathrm{H}^{-1/2}(\partial\Omega_j)$ (so-called Dirichlet and Neumann traces) uniquely defined by

$$\gamma^j_{\mathrm{D}}(\varphi) := \varphi|_{\partial\Omega_j} \quad \text{and} \quad \gamma^j_{\mathrm{N}}(\varphi) := \mathbf{n}_j \cdot \nabla\varphi|_{\partial\Omega_j} \qquad \forall \varphi \in \mathscr{C}^\infty(\overline{\Omega}_j).$$

In the definition above $\mathbf{n}_j$ refers to the vector field normal to $\partial\Omega_j$ pointing toward the exterior of Ω_j. We will also need a notation to refer to an operator gathering both traces in a single array

$$\gamma^j(u) := (\gamma^j_{\mathrm{D}}(v), \gamma^j_{\mathrm{N}}(v))$$

We shall also refer to $\gamma^j_{\mathrm{D},c}, \gamma^j_{\mathrm{N},c}$ defined in the same manner as $\gamma^j_{\mathrm{D}}, \gamma^j_{\mathrm{N}}$ with traces taken from the exterior of Ω_j. In addition, we set $\gamma^j_c(v) := (\gamma^j_{\mathrm{D},c}(v), \gamma^j_{\mathrm{N},c}(v))$. We will refer to mean values and jumps to these trace operators, setting

$$\{\gamma^j(u)\} := \frac{1}{2}\big(\gamma^j(u) + \gamma^j_c(u)\big) \quad \text{and} \quad [\gamma^j(u)] := \gamma^j(u) - \gamma^j_c(u)$$

2 Trace spaces

We want to recast Problem (2) into variational boundary integral equations set in trace spaces adapted to the present multi-sub-domain context. The most simple space we can introduce consists in the *multi-trace space* [3, Sect. 2.1] i.e. the cartesian product of local traces:

$$\begin{aligned}&\mathbb{H}(\Sigma) \;:=\; \mathbb{H}(\Gamma_0)\times\cdots\times\mathbb{H}(\Gamma_n)\\&\text{where}\quad \mathbb{H}(\Gamma_j):=\mathrm{H}^{+\frac{1}{2}}(\Gamma_j)\times\mathrm{H}^{-\frac{1}{2}}(\Gamma_j)\;.\end{aligned}\tag{3}$$

We endow each $\mathbb{H}(\Gamma_j)$ with the norm $\|(v,q)\|_{\mathbb{H}(\Gamma_j)} := (\|v\|^2_{\mathrm{H}^{1/2}(\Gamma_j)}+\|q\|^2_{\mathrm{H}^{-1/2}(\Gamma_j)})^{1/2}$, and equip $\mathbb{H}(\Sigma)$ with the norm naturally associated with the cartesian product

$$\|\mathfrak{u}\|_{\mathbb{H}(\Sigma)} := \left(\, \|\mathfrak{u}_0\|^2_{\mathbb{H}(\Gamma_0)}+\cdots+\|\mathfrak{u}_n\|^2_{\mathbb{H}(\Gamma_n)}\,\right)^{\frac{1}{2}}$$

for $\mathfrak{u}=(\mathfrak{u}_0,\dots,\mathfrak{u}_n)\in\mathbb{H}(\Sigma)$[‡]. In the sequel we shall repeatedly refer to the continuous operator $\gamma:\Pi_{j=0}^n\mathrm{H}^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_j)\to\mathbb{H}(\Sigma)$ defined by $\gamma(u):=(\gamma_0(u),\dots,\gamma_n(u))$, where $\Pi_{j=0}^n\mathrm{H}^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_j)$ should be understood as the set of $u\in\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that $u|_{\Omega_j}\in\mathrm{H}^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_j)$ for all j. We also need a bilinear duality pairing for $\mathbb{H}(\Gamma_j)$ and $\mathbb{H}(\Sigma)$; writing $\langle\,,\,\rangle_{\Gamma_j}$ for the duality pairing between $\mathrm{H}^{1/2}(\Gamma_j)$ and $\mathrm{H}^{-1/2}(\Gamma_j)$, we opt for the skew-symmetric bilinear form

$$\begin{aligned}&[\![\mathfrak{u},\mathfrak{v}]\!] := \sum_{j=0}^n\,[\mathfrak{u}_j,\mathfrak{v}_j]_{\Gamma_j}\qquad\text{where}\\&\left[\begin{pmatrix}u_j\\p_j\end{pmatrix},\begin{pmatrix}v_j\\q_j\end{pmatrix}\right]_{\Gamma_j} \;:=\; \langle u_j,q_j\rangle_{\Gamma_j}-\langle v_j,p_j\rangle_{\Gamma_j}\;.\end{aligned}\tag{4}$$

Next, as in [3, Sect. 2.2], [5, Sect. 3.1], we introduce the so-called single-trace space that consists in collections of traces complying with transmission conditions. This space can be defined by

$$\mathbb{X}(\Sigma) := \mathrm{clos}_{\,\mathbb{H}(\Sigma)}\{\,\gamma(u)=(\gamma^j(u))_{j=0}^n \mid u\in\mathscr{C}^\infty(\mathbb{R}^d)\,\}\tag{5}$$

where the symbol $\mathrm{clos}_{\,\mathbb{H}(\Sigma)}$ refers to the closure with respect to the norm on $\mathbb{H}(\Sigma)$. By construction, this is a closed subspace of $\mathbb{H}(\Sigma)$. Note also that a function $u\in\mathrm{H}^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_0)\times\cdots\times\mathrm{H}^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_n)$ satisfies the transmission conditions of (2), if and only if $\gamma(u)=(\gamma^j(u))_{j=0}^n\in\mathbb{X}(\Sigma)$. In particular, if $u\in\mathrm{H}^1_{\mathrm{loc}}(\Delta,\mathbb{R}^d)$ then $\gamma(u)=(\gamma^j(u))_{j=0}^n\in\mathbb{X}(\Gamma)$. In the sequel, we will use this space to enforce transmission conditions. The single-trace space admits a simple weak characterization, see [3, Prop.2.1].

Lemma 2.1.
For any $\mathfrak{u}\in\mathbb{H}(\Sigma)$ *we have* $\mathfrak{u}\in\mathbb{X}(\Sigma)\iff[\![\mathfrak{u},\mathfrak{v}]\!]=0\;\forall\mathfrak{v}\in\mathbb{X}(\Sigma)$.

3 Summary of potential theory

In this paragraph, we shall remind the reader of well established results concerning the integral representation of solutions to homogeneous Helmholtz equation in Lipshitz domains. A

[‡]Functions in Dirichlet trace spaces like $\mathrm{H}^{1/2}(\partial\Omega_j)$ will be denoted by u,v,w, whereas we use p,q,r for Neumann traces. Small fraktur font symbols $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ are reserved for Cauchy traces, with an integer subscript indicating restriction to a sub-domain boundary.

detailed proof of the statements contained in the present paragraph can be found for example in [22, Chap.3].

Let $\mathscr{G}_\kappa(\boldsymbol{x})$ refer to the outgoing Green's kernel associated to the Helmholtz operator $-\Delta-\kappa^2$. For example $\mathscr{G}_\kappa(\boldsymbol{x}) = \exp(i\kappa|\boldsymbol{x}|)/(4\pi|\boldsymbol{x}|)$ in $\mathbb{R}^3$. For each sub-domain Ω_j, any $(v,q)\in\mathbb{H}(\Gamma_j)$ and any $\boldsymbol{x}\in\mathbb{R}^d\setminus\Gamma_j$, define the potential operator

$$\mathrm{G}_\kappa^j(v,q)(\boldsymbol{x}) := \int_{\Gamma_j} q(\boldsymbol{y})\,\mathscr{G}_\kappa(\boldsymbol{x}-\boldsymbol{y}) + v(\boldsymbol{y})\,\mathbf{n}_j(\boldsymbol{y})\cdot(\nabla\mathscr{G}_\kappa)(\boldsymbol{x}-\boldsymbol{y})d\sigma(\boldsymbol{y})\;. \tag{6}$$

The operator G_κ^j maps continuously $\mathbb{H}(\Gamma_j)$ into $\mathrm{H}^1(\Delta,\overline{\Omega}_j)\times\mathrm{H}^1(\Delta,\mathbb{R}^d\setminus\Omega_j)$, see [22, Thm 3.1.16]. In particular G_κ^j can be applied to a pair of traces of the form $\mathfrak{v}=\gamma^j(v)$. This potential operator can be used to write a representation formula for solution to homogeneous Helmholtz equations, see [22, Thm 3.1.6].

Proposition 3.1.
Let $u\in\mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}_j)$ satisfy $\Delta u+\kappa_j^2 u=0$ in Ω_j. Assume in addition that u is κ_0-outgoing if $j=0$. We have the representation formula

$$\mathrm{G}_{\kappa_j}^j(\gamma^j(u))(\boldsymbol{x}) \;=\; \begin{cases} u(\boldsymbol{x}) & \text{for } \boldsymbol{x}\in\Omega_j\\ 0 & \text{for } \boldsymbol{x}\in\mathbb{R}^d\setminus\overline{\Omega}_j\;. \end{cases} \tag{7}$$

Similarly, if $v\in\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\Omega_j)$ satisfies $\Delta v+\kappa_j^2 v=0$ in $\mathbb{R}^d\setminus\overline{\Omega}_j$, and is κ_j-outgoing radiating in the case $j\neq 0$, then we have $\mathrm{G}_{\kappa_j}^j(\gamma_c^j(v))(\boldsymbol{x})=-v(\boldsymbol{x})$ for $\boldsymbol{x}\in\mathbb{R}^d\setminus\overline{\Omega}_j$, and $\mathrm{G}_{\kappa_j}^j(\gamma_c^j(v))(\boldsymbol{x})=0$ for $\boldsymbol{x}\in\Omega_j$.

The potential operator (6) also satisfies a remarkable identity, known as jump formula, describing the behavior of $\mathrm{G}_{\kappa_j}^j(\mathfrak{v})(\boldsymbol{x})$ as $\boldsymbol{x}$ crosses Γ_j, namely

$$[\gamma^j]\cdot\mathrm{G}_{\kappa_j}^j(\mathfrak{v})=\mathfrak{v}\qquad\forall\mathfrak{v}\in\mathbb{H}(\Gamma_j) \tag{8}$$

which also writes $[\gamma^j]\cdot\mathrm{G}_{\kappa_j}^j=\mathrm{Id}$. We refer the reader to [22, Thm.3.3.1] (the jump formulas are more commonly written in the form of four equations in the literature). Proposition 3.1 shows that, if u is solution to a homogeneous Helmholtz equation in Ω_j (and is outgoing if $j=0$) then $\gamma^j\cdot\mathrm{G}_{\kappa_j}^j(\gamma^j(u))=\gamma^j(u)$. This actually turns out to be a characterization of traces of solutions to homogeneous Helmholtz equation.

Proposition 3.2.
Define $\mathcal{C}_\kappa^{\mathrm{in}}(\Omega_j):=\{\gamma^j(u)\mid u\in\mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}_j)\;,\;\Delta u+\kappa^2u=0$ in $\Omega_j,\; u$ outgoing if $j=0\;\}$. Then $\gamma^j\cdot\mathrm{G}_\kappa^j:\mathbb{H}(\Gamma_j)\to\mathbb{H}(\Gamma_j)$ is a continuous projector, called Calderón projector interior to Ω_j, whose range coincides with $\mathcal{C}_\kappa^{\mathrm{in}}(\Omega_j)$ i.e. for any $\mathfrak{v}\in\mathbb{H}(\Gamma_j)$

$$\gamma^j\cdot\mathrm{G}_\kappa^j(\mathfrak{v})=\mathfrak{v}\qquad\Longleftrightarrow\qquad\mathfrak{v}\in\mathcal{C}_\kappa^{\mathrm{in}}(\Omega_j)\;.$$

Similarly, defining $\mathcal{C}_\kappa^{\mathrm{out}}(\Omega_j):=\{\gamma_c^j(u)\mid u\in\mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}_j),\;\Delta u+\kappa^2u=0$ in $\mathbb{R}^d\setminus\Omega_j,\; u$ outgoing if $j\neq 0\;\}$, we have $\gamma^j\cdot\mathrm{G}_\kappa^j(\mathfrak{v})=0$ if and only if $\mathfrak{v}\in\mathcal{C}_\kappa^{\mathrm{out}}(\Omega_j)$.

For a detailed proof, see [22, Prop.3.6.2]. We shall repeatedly use this characterization as a convenient way to express wave equations in the sub-domains Ω_j. Here is another characterization of the space of Cauchy data which was established in [3, Lemma 6.2].

Lemma 3.1.
Consider any $j = 0, \dots n$, and any $\kappa \in \mathbb{C} \setminus \{0\}$ such that $\Re e\{\kappa\} \geq 0$, $\Im m\{\kappa\} \geq 0$. Then for any $\mathfrak{u} \in \mathbb{H}(\Gamma_j)$ we have: $\mathfrak{u} \in \mathcal{C}^{\mathrm{in}}_{\kappa}(\Omega_j) \iff [\mathfrak{u}, \mathfrak{v}]_{\Gamma_j} = 0 \ \forall \mathfrak{v} \in \mathcal{C}^{\mathrm{in}}_{\kappa}(\Omega_j)$. Similarly we have $\mathfrak{u} \in \mathcal{C}^{\mathrm{out}}_{\kappa}(\Omega_j) \iff [\mathfrak{u}, \mathfrak{v}]_{\Gamma_j} = 0 \ \forall \mathfrak{v} \in \mathcal{C}^{\mathrm{out}}_{\kappa}(\Omega_j)$.

The results that we have stated above hold for any $j = 0 \dots n$. We also set $\mathcal{C}^{\alpha}(\Sigma) := \mathcal{C}^{\alpha}_{\kappa_0}(\Omega_0) \times \cdots \times \mathcal{C}^{\alpha}_{\kappa_n}(\Omega_n)$ for $\alpha = \mathrm{in}, \mathrm{out}$. The notations just introduced allow a condensed reformulation of the well-posedness of (2), see [3, Prop.6.1] for a detailed discussion and proof.

Lemma 3.2.
$\mathbb{X}(\Sigma) \oplus \mathcal{C}^{\mathrm{in}}(\Sigma) = \mathbb{H}(\Sigma)$.

To handle Calderón projectors in a multi-sub-domain context, it is more comfortable to introduce global operators, so as to reduce notations. First we introduce the continuous operator $\mathrm{A} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ defined by

$$\begin{aligned} &[\![\mathrm{A}(\mathfrak{u}), \mathfrak{v}]\!] := \textstyle\sum_{j=0}^{n} \left[\mathrm{A}^j(\mathfrak{u}_j), \mathfrak{v}_j)\right]_{\Gamma_j} \\ &\text{with} \quad \mathrm{A}^j := 2\,\{\gamma^j\} \cdot \mathrm{G}^j_{\kappa_j} \end{aligned} \tag{9}$$

For all $\mathfrak{u} = (\mathfrak{u}_j)_{j=0}^n, \mathfrak{v} = (\mathfrak{v}_j)_{j=0}^n \in \mathbb{H}(\Sigma)$. Observe that $(\mathrm{Id} \pm \mathrm{A})/2$ are projectors, according to Proposition 3.2, with $\ker(\mathrm{A} - \mathrm{Id}) = \mathrm{range}(\mathrm{A} + \mathrm{Id}) = \mathcal{C}^{\mathrm{in}}(\Sigma)$ and $\ker(\mathrm{A} + \mathrm{Id}) = \mathrm{range}(\mathrm{A} - \mathrm{Id}) = \mathcal{C}^{\mathrm{out}}(\Sigma)$. With this notation, a direct consequence of Proposition 3.2 is that $(\mathrm{A}^j)^2 = (2\{\gamma^j\} \cdot \mathrm{G}^j_{\kappa})^2 = \mathrm{Id}$. This is summarized in the next lemma.

Lemma 3.3.
$(\mathrm{A})^2 = \mathrm{Id}$.

Remark 3.1. *In the case of two sub-domains $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1$ and one interface $\Sigma = \Gamma_0 = \Gamma_1$, with $\kappa = \kappa_0 = \kappa_1$, there is a remarkable identity relating A^0 to A^1. Indeed, in this situation, the only difference in the definition of G^0_{κ} and G^1_{κ} comes from $\mathbf{n}_0 = -\mathbf{n}_1$. In particular, denoting*

$$Q := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

we have $\mathrm{G}^0_{\kappa}(\mathfrak{u}) = -\mathrm{G}^1_{\kappa}(Q \cdot \mathfrak{u}) \ \forall \mathfrak{u} \in \mathrm{H}^{1/2}(\Gamma_0) \times \mathrm{H}^{-1/2}(\Gamma_0)$. Note that $\{\gamma^1\} = Q \cdot \{\gamma^0\}$, so multiplying the previous equality by $\{\gamma^1\}$ yields $Q \cdot \mathrm{A}^0 = -\mathrm{A}^1 \cdot Q$. □

Lemma 3.3 above shows directly that A is invertible. It also satisfies a generalized Garding inequality. The next result is proved for example in [20, §4].

Proposition 3.3.
Define $\Theta : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ by $\Theta(\mathfrak{u}) := (Q(\mathfrak{u}_0), Q(\mathfrak{u}_1), \dots, Q(\mathfrak{u}_n))$ for all $\mathfrak{u} = (\mathfrak{u}_0, \mathfrak{u}_1, \dots, \mathfrak{u}_n)$, where $Q(u, p) := (u, -p)$ for any $(u, p) \in \mathrm{H}^{1/2}(\Gamma_j) \times \mathrm{H}^{-1/2}(\Gamma_j)$ and any j. There exists a compact operator $\mathrm{K} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ and a constant $C > 0$ such that

$$\Re e\{-[\![\mathrm{A}\mathfrak{u}, \Theta(\overline{\mathfrak{u}})]\!] + [\![\mathrm{K}\mathfrak{u}, \overline{\mathfrak{u}}]\!]\} \geq C\,\|\mathfrak{u}\|^2_{\mathbb{H}(\Sigma)} \qquad \forall \mathfrak{u} \in \mathbb{H}(\Sigma).$$

4 Local multi-trace formulation

In this section we would like to recall the derivation of the local multi-trace formulation introduced in [11], and provide detailed analysis for it. The formulation considered in [10] is a particular case of the formulations introduced in [11]. But only an analysis of the formulation of [10] has been provided so far. In addition, we should underline our hypothesis (1) that discards any junction point in the sub-domain partition, while [10, 11] also considered geometrical configurations with junction points.

A key ingredient of local multi-trace theory is an operator yielding a characterization of transmission conditions of (2). Considering $\mathfrak{u} = (u_k, p_k)_{k=0}^n$ and $\mathfrak{v} = (v_j, q_j)_{j=0}^n$, we define the transmission operator $\Pi : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ by

$$\mathfrak{v} = \Pi(\mathfrak{u}) \quad \Longleftrightarrow \quad \begin{cases} v_j = +\,u_k \\ q_j = -\,p_k \end{cases} \quad \text{on } \Gamma_{j,k} \quad \forall j,k = 0\ldots n, \;\; j \neq k. \tag{10}$$

Clearly, for any function $u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$ we have $\gamma(u) := (\gamma^j(u))_{j=0}^n \in \mathbb{H}(\Sigma)$ with $\gamma(u) = \Pi(\gamma(u))$. Conversely, considering any function $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that $u|_{\Omega_j} \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j)$ for all $j = 0\ldots n$, then $\gamma(u) \in \mathbb{H}(\Sigma)$ is well defined, and if $\gamma(u) = \Pi(\gamma(u))$ then $u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$. Routine calculus shows the following remarkable identities

$$\Pi^2 = \mathrm{Id}\ , \quad \overline{\Pi(\mathfrak{v})} = \Pi(\overline{\mathfrak{v}}) \quad \text{and} \quad [\![\Pi(\mathfrak{u}), \mathfrak{v}]\!] = [\![\Pi(\mathfrak{v}), \mathfrak{u}]\!] \quad \forall \mathfrak{u}, \mathfrak{v} \in \mathbb{H}(\Sigma). \tag{11}$$

As is readily checked, the operator Π maps continuously $\mathbb{H}(\Sigma)$ onto $\mathbb{H}(\Sigma)$ *under Assumption (1) that each* $\Gamma_{j,k}$ *is a Lipschitz manifold without boundary.* Elementary arguments on trace spaces show that, for any $\mathfrak{u} \in \mathbb{H}(\Sigma)$, we have $\mathfrak{u} \in \mathbb{X}(\Sigma) \iff \mathfrak{u} = \Pi(\mathfrak{u})$. Since $\Pi^2 = \mathrm{Id}$, this can be simply rewritten in the following manner.

Lemma 4.1.
$\mathrm{range}(\Pi + \mathrm{Id}) = \mathbb{X}(\Sigma)$.

Now consider $\mathfrak{u} = (\gamma^j(u))_{j=0}^n$ the traces of the unique solution u to Problem (2). The homogeneous wave equation satisfied by u in each sub-domain can be reformulated by means of Calderón projectors $(\mathrm{A} - \mathrm{Id})(\mathfrak{u} - \mathfrak{u}_{\mathrm{inc}}) = 0$. Choose any relaxation parameter $\alpha \in \mathbb{C} \setminus \{0\}$ and add $\alpha\,(\mathrm{Id} - \Pi)\mathfrak{u} = 0$ to this equation, which is consistent since $\mathfrak{u} = \gamma(u)$ must satisfy the transmission conditions of (2). Denoting $\mathfrak{f} := (\mathrm{A} - \mathrm{Id})\mathfrak{u}_{\mathrm{inc}}$, what precedes implies

$$\begin{cases} \mathfrak{u} \in \mathbb{H}(\Sigma) \quad \text{and} \\ [\![\, (\mathrm{A} - \Pi_\alpha)\mathfrak{u}, \mathfrak{v} \,]\!] = [\![\mathfrak{f}, \mathfrak{v}]\!] \quad \forall \mathfrak{v} \in \mathbb{H}(\Sigma). \end{cases} \tag{12}$$
$$\text{where} \quad \Pi_\alpha := (1 - \alpha)\mathrm{Id} + \alpha\Pi$$

Observe that the operator of the formulation above can also be rewritten in the form of a convex combination $\mathrm{A} - \Pi_\alpha = (1 - \alpha)(\mathrm{A} - \mathrm{Id}) + \alpha(\mathrm{A} - \Pi)$. Existence and uniqueness of the solution to this formulation has already been established only in the case $\alpha = 1$, see [10, Thm.9 and Thm.11]. For all other values of α, well-posedness of this formulation was an open problem so far, as mentioned in [11, Rem.2]. Below we prove uniqueness of the solution to (12) for *any* $\alpha \in \mathbb{C} \setminus \{0\}$.

Proposition 4.1.
$\mathrm{Ker}(\mathrm{A}-\Pi_\alpha)=\{0\}\quad\forall\alpha\in\mathbb{C}\setminus\{0\}.$

Proof:
Take $\mathfrak{u}=(\mathfrak{u}_j)_{j=0}^n$ satisfying $(\mathrm{A}-\mathrm{Id})\mathfrak{u}+\alpha(\mathrm{Id}-\Pi)\mathfrak{u}=0$. Thus we have $\mathfrak{w}:=\alpha(\Pi-\mathrm{Id})\mathfrak{u}=(\mathrm{A}-\mathrm{Id})\mathfrak{u}\in\mathrm{range}(\mathrm{A}-\mathrm{Id})\cap\mathrm{range}(\Pi-\mathrm{Id})$. Denote $\mathfrak{w}_j$ the component of $\mathfrak{w}$ associated to $\partial\Omega_j$ so that $\mathfrak{w}=(\mathfrak{w}_0,\cdots,\mathfrak{w}_n)$, and set $\psi_j(\boldsymbol{x}):=\mathrm{G}^j_{\kappa_j}(\mathfrak{w}_j)(\boldsymbol{x})$. We have $(\mathrm{A}+\mathrm{Id})\mathfrak{w}=(\mathrm{A}^2-\mathrm{Id})\mathfrak{u}=0$, which can be rewritten $\gamma^j\cdot\mathrm{G}^j_{\kappa_j}(\mathfrak{w}_j)=\gamma^j(\psi_j)=0$ for all $j=0\dots n$. Since we also have, by construction, $-\Delta\psi_j-\kappa_j^2\psi_j=0$ in Ω_j, we conclude that $\psi_j=0$ in Ω_j, and $\mathfrak{w}_j=[\gamma^j]\cdot\mathrm{G}^j_{\kappa_j}(\mathfrak{w}_j)=[\gamma^j(\psi_j)]=-\gamma^j_c(\psi_j)$. If we can prove that $\psi_j=0$ in $\mathbb{R}^d\setminus\overline{\Omega}_j$ for each j, this will show that $\mathfrak{w}=0$.

Now observe that, since $\mathfrak{w}\in\mathrm{range}(\Pi-\mathrm{Id})$ we have $\Pi(\mathfrak{w})+\mathfrak{w}=0$ i.e $\Pi(\mathfrak{w})=-\mathfrak{w}$. As a consequence the functions ψ_j satisfy an homogeneous problem with "anti-transmission conditions"

$$\left\{\begin{aligned}&-\Delta\psi_j-\kappa_j^2\psi_j=0\quad\text{in }\mathbb{R}^d\setminus\overline{\Omega}_j\quad\forall j=0\dots n\,,\\&\psi_j\text{ is outgoing (with respect to }\kappa_j)\text{ for }j\neq 0\\&\gamma^j_{\mathrm{D},c}(\psi_j)+\gamma^k_{\mathrm{D},c}(\psi_k)=0\quad\text{and}\\&\gamma^j_{\mathrm{N},c}(\psi_j)-\gamma^k_{\mathrm{N},c}(\psi_k)=0\quad\text{on }\Gamma_j\cap\Gamma_k\quad\forall j,k.\end{aligned}\right.\tag{13}$$

Since $\Pi^2=\mathrm{Id}$ and $\Pi(\mathfrak{w})=-\mathfrak{w}$, we have $2\mathfrak{w}=\mathfrak{w}-\Pi(\mathfrak{w})$, and $\overline{\mathfrak{w}}+\Pi(\overline{\mathfrak{w}})=0$. From this and (11), we obtain $2[\![\mathfrak{w},\overline{\mathfrak{w}}]\!]=[\![\mathfrak{w}-\Pi(\mathfrak{w}),\overline{\mathfrak{w}}]\!]=-[\![\overline{\mathfrak{w}}+\Pi(\overline{\mathfrak{w}}),\mathfrak{w}]\!]=0$. This can be rewritten

$$\sum_{j=0}^n\Im m\Big\{\int_{\Gamma_j}\gamma^j_{\mathrm{D},c}(\psi_j)\gamma^j_{\mathrm{N},c}(\overline{\psi}_j)d\sigma\Big\}=0.\tag{14}$$

Take $r>0$ sufficiently large to garantee that $\mathbb{R}^d\setminus\Omega_0\subset\mathrm{B}_r$ where $\mathrm{B}_r\subset\mathbb{R}^d$ refers to the ball centered at 0 with radius r. Since $-\Delta\psi_j-\kappa_j^2\psi_j=0$ in $\mathbb{R}^d\setminus\overline{\Omega}_j$, applying Green's formula in each $\mathrm{B}_r\setminus\overline{\Omega}_j$ yields

$$\begin{aligned}\int_{\partial\mathrm{B}_r}\psi_j\partial_r\overline{\psi}_jd\sigma&=\int_{\mathrm{B}_r\setminus\Omega_j}|\nabla\psi_j|^2-\kappa_j^2|\psi_j|^2d\boldsymbol{x}+\int_{\partial\Omega_j}\gamma^j_{\mathrm{D},c}(\psi_j)\gamma^j_{\mathrm{N},c}(\overline{\psi}_j)d\sigma\\0&=\int_{\mathrm{B}_r\setminus\Omega_0}|\nabla\psi_0|^2-\kappa_0^2|\psi_0|^2d\boldsymbol{x}+\int_{\partial\Omega_0}\gamma^0_{\mathrm{D},c}(\psi_0)\gamma^0_{\mathrm{N},c}(\overline{\psi}_0)d\sigma\end{aligned}$$

In these equations, "∂_r" refers to the radial derivative. Take the imaginary part of the identities above, and sum over $j=0\dots n$, taking account of (14). This leads to

$$\sum_{j=0}^n\Im m\Big\{\int_{\partial\mathrm{B}_r}\psi_j\partial_r\overline{\psi}_jd\sigma\Big\}=\Im m\Big\{\sum_{j=0}^n\int_{\partial\Omega_j}\gamma^j_{\mathrm{D},c}(\psi_j)\gamma^j_{\mathrm{N},c}(\overline{\psi}_j)d\sigma\Big\}=\frac{1}{2\imath}[\![\mathfrak{w},\overline{\mathfrak{w}}]\!]=0$$

where $\imath$ refers to the imaginary unit. By construction, the functions ψ_j are outgoing radiating, so that $0=\lim_{r\to\infty}\int_{\partial\mathrm{B}_r}|\partial_r\psi_j-\imath\kappa_j\psi_j|^2d\sigma=0$. As a consequence, we finally obtain

$$\sum_{j=0}^{n}\frac{1}{\kappa_j}\int_{\partial \mathrm{B}_r}|\partial_r\psi_j|^2+\kappa_j^2|\psi_j|^2 d\sigma$$
$$=\sum_{j=1}^{n}\frac{1}{\kappa_j}\int_{\partial \mathrm{B}_r}|\partial_r\psi_j-i\kappa_j\psi_j|^2 d\sigma+2\sum_{j=1}^{n}\Im m\Big\{\int_{\partial \mathrm{B}_r}\psi_j\partial_r\overline{\psi}_j d\sigma\Big\}$$
$$=\sum_{j=1}^{n}\frac{1}{\kappa_j}\int_{\partial \mathrm{B}_r}|\partial_r\psi_j-i\kappa_j\psi_j|^2 d\sigma \underset{r\to\infty}{\longrightarrow} 0$$

This shows in particular that $\lim_{r\to\infty}\int_{\partial \mathrm{B}_r}|\psi_j|^2 d\sigma = 0$ for all $j = 1\dots n$. As a consequence we can apply Rellich's lemma, see [6, Lemma 3.11], which implies that $\psi_j = 0$ in the unbounded connected component of each $\mathbb{R}^d\setminus\overline{\Omega}_j$.

Let us show that ψ_j also vanishes in bounded connected components of $\mathbb{R}^d\setminus\overline{\Omega}_j$. Take an arbitrary j, and let $\mathcal{O}$ be a bounded connected component of $\mathbb{R}^d\setminus\overline{\Omega}_j$. We have $\partial\mathcal{O} = \partial\Omega_j\cap\partial\Omega_k = \Gamma_{j,k}$ for some $k = 0\dots n, k\neq j$. Let $\mathcal{O}'$ be the unbounded connected component of $\mathbb{R}^d\setminus\overline{\Omega}_k$. Then we have $\Omega_j\subset\mathcal{O}'$, and $\partial\mathcal{O}' = \partial\mathcal{O} = \partial\Omega_j\cap\partial\Omega_k$. Since $\psi_k = 0$ in $\mathcal{O}'$, we have $\gamma^j_{\mathrm{D},c}(\psi_j)|_{\Gamma_{j,k}} = -\gamma^k_{\mathrm{D},c}(\psi_k) = 0$ and $\gamma^j_{\mathrm{N},c}(\psi_j)|_{\Gamma_{j,k}} = \gamma^k_{\mathrm{N},c}(\psi_k) = 0$, according to the transmission conditions of (13). Finally we have $-\Delta\psi_j-\kappa_j^2\psi_j = 0$ in $\mathcal{O}$ with $\gamma^j_c(\psi_j) = 0$ on $\partial\mathcal{O}$. We conclude by unique continuation principle (see [15, §.4.3]) that $\psi_j = 0$ in $\mathcal{O}$. We have just proved that

$$\psi_j = 0 \quad \text{in } \mathbb{R}^d\setminus\overline{\Omega}_j \quad \forall j = 0\dots n.$$

Finally we have, on the one hand, $\mathfrak{w} = (\mathrm{A}-\mathrm{Id})\mathfrak{u} = 0$, so $\mathfrak{u} = (\mathrm{A}+\mathrm{Id})\mathfrak{u}/2 \in \mathrm{range}(\mathrm{A}+\mathrm{Id}) = \mathcal{C}^{\mathrm{in}}(\Sigma)$, and on the other hand $\mathfrak{w} = \alpha(\Pi-\mathrm{Id})\mathfrak{u} = 0$, so $\mathfrak{u} = (\Pi+\mathrm{Id})\mathfrak{u}/2 \in \mathrm{range}(\Pi+\mathrm{Id}) = \mathbb{X}(\Sigma)$. So we conclude that $\mathfrak{u}\in\mathcal{C}^{\mathrm{in}}(\Sigma)\cap\mathbb{X}(\Sigma) = \{0\}$ according to Lemma 3.2. □

5 Examples

Before going further into the analysis of the local multi-trace formulation (12), we dedicate this section to deriving and studying it in ultra simplified situations where all calculations can be conducted quasi-explicitly. Here we will systematically consider the case where all wave numbers are equal

$$\kappa_0 = \kappa_1 = \cdots = \kappa_n. \tag{15}$$

This assumption will allow substantial simplifications. Another purpose of the present section is to determine the spectrum of the multi-trace operator in these simplified situations.

5.1 Two domain configuration

We start by considering the case where the space is partitioned in two domains only. This simple case was already considered in [10, §3.1], but here we are going to formulate additional comments. In this case $\Sigma = \Gamma_0 = \Gamma_1 = \Gamma_{0,1}$. We want to represent the operator $(1-\alpha)(\mathrm{A}-\mathrm{Id})+\alpha(\mathrm{A}-\Pi) : \mathbb{H}(\Gamma_{0,1})\times\mathbb{H}(\Gamma_{0,1})\to\mathbb{H}(\Gamma_{0,1})\times\mathbb{H}(\Gamma_{0,1})$ in a matrix form. First of all, note that the operator Π admits the following expression,

$$\Pi\left(\begin{bmatrix}\mathfrak{v}_0\\ \mathfrak{v}_1\end{bmatrix}\right) = \begin{bmatrix}0 & \mathrm{Q}\\ \mathrm{Q} & 0\end{bmatrix}\cdot\begin{bmatrix}\mathfrak{v}_0\\ \mathfrak{v}_1\end{bmatrix} \qquad \text{with} \qquad Q := \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{16}$$

Hence, denoting $\mathfrak{u} = (\mathfrak{u}_0, \mathfrak{u}_1)$, we have

$$(\mathrm{A} - \alpha\Pi)\mathfrak{u} \;=\; \begin{bmatrix} \mathrm{A}^0 & -\alpha\mathrm{Q} \\ -\alpha\mathrm{Q} & \mathrm{A}^1 \end{bmatrix} \cdot \begin{bmatrix} \mathfrak{u}_0 \\ \mathfrak{u}_1 \end{bmatrix}.$$

To determine the spectrum of the (relaxed) multi-trace operator $(1-\alpha)(\mathrm{A}-\mathrm{Id})+\alpha(\mathrm{A}-\Pi)$, it suffices to determine the spectrum of $\mathrm{A}-\alpha\Pi$. If we compute the square of this operator, taking account of (15), we obtain $(\mathrm{A}-\alpha\Pi)^2 = \mathrm{A}^2+\alpha^2\Pi^2-\alpha(\Pi\mathrm{A}+\mathrm{A}\Pi) = (1+\alpha^2)\mathrm{Id}-\alpha(\Pi\mathrm{A}+\mathrm{A}\Pi)$. In this case, a direct calculus shows that $\mathrm{Q}\mathrm{A}^0 = -\mathrm{A}^1\mathrm{Q}$. As a consequence, an explicit calculus yields

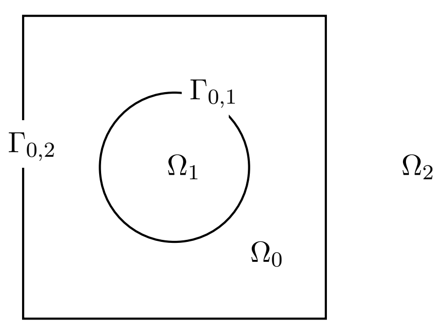
$$\begin{aligned} \begin{bmatrix} \mathrm{A}^0 & 0 \\ 0 & \mathrm{A}^1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \mathrm{Q} \\ \mathrm{Q} & 0 \end{bmatrix} &= \begin{bmatrix} 0 & \mathrm{A}^0\mathrm{Q} \\ \mathrm{A}^1\mathrm{Q} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\mathrm{Q}\mathrm{A}^1 \\ -\mathrm{Q}\mathrm{A}^0 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & \mathrm{Q} \\ \mathrm{Q} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{A}^0 & 0 \\ 0 & \mathrm{A}^1 \end{bmatrix} \end{aligned}$$

From this we conclude that $\Pi\mathrm{A} + \mathrm{A}\Pi = 0$, which finally yields $(\mathrm{A} - \alpha\Pi)^2 = (1 + \alpha^2)\,\mathrm{Id}$. This expression, together with the spectral mapping theorem (see [21, Thm.10.28] for example), provides an explicit characterization of the spectrum in the case where all wave-numbers are equal[§]

$$\mathfrak{S}(\mathrm{A} - \Pi_\alpha) \subset \{-1 + \alpha + \sqrt{1 + \alpha^2}, -1 + \alpha - \sqrt{1 + \alpha^2}\}.$$

5.2 Three domain configuration

Now we consider a partition in three domains $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \overline{\Omega}_2$. This situation is pictured below.



As the definition of the transmission operator is given by a formula written on each interface, let us decompose traces on each sub-domain according to interfaces. Considering the decomposition $\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2}$, any trace $\mathfrak{v} \in \mathbb{H}(\Gamma_0)$ induces an element $\mathrm{R}^0_1(\mathfrak{v}) \in \mathbb{H}(\Gamma_{0,1})$ defined by $\mathrm{R}^0_1(\mathfrak{v}) = \mathfrak{v}|_{\Gamma_{0,1}}$. We may define $\mathrm{R}^0_2(\mathfrak{v}) \in \mathbb{H}(\Gamma_{0,2})$ similarly. This establishes a natural isomorphism $(\mathrm{R}^0_1, \mathrm{R}^0_2) : \mathbb{H}(\Gamma_0) \to \mathbb{H}(\Gamma_{0,1}) \times \mathbb{H}(\Gamma_{0,2})$. The adjoint of those maps are extension operators i.e. $(\mathrm{R}^0_j)^*(\mathfrak{v}) = \mathfrak{v} \cdot 1_{\Gamma_{0,j}} \in \mathbb{H}(\Gamma_0)$ for any $\mathfrak{v} \in \mathbb{H}(\Gamma_{0,j})$. With these maps, the operator

[§]In the remaining of this article, the square root of complex numbers shall be defined by $\sqrt{\rho \exp(i\theta)} = \sqrt{\rho}\exp(i\theta/2)$ for $\theta \in (-\pi, \pi]$.

A^0 induces a 2×2 matrix denoted $[A^0]$ with integral operator entries

$$[A^0] = \begin{bmatrix} A^0_{1,1} & A^0_{1,2} \\ A^0_{2,1} & A^0_{2,2} \end{bmatrix} \qquad \text{with} \quad A^0_{j,k} := R^0_j \cdot A^0 \cdot (R^0_k)^*$$

Plugging this decomposition into the definition of the operator A yields a 4×4 matrix of integral operators acting on tuples $(\mathfrak{u}_{0,1}, \mathfrak{u}_{0,2}, \mathfrak{u}_1, \mathfrak{u}_2) \in \mathbb{H}(\Gamma_{0,1}) \times \mathbb{H}(\Gamma_{0,2}) \times \mathbb{H}(\Gamma_1) \times \mathbb{H}(\Gamma_2)$, and given by the following formula

$$(A - \alpha\Pi)\mathfrak{u} = \begin{bmatrix} A^0_{1,1} & A^0_{1,2} & -\alpha Q & 0 \\ A^0_{2,1} & A^0_{2,2} & 0 & -\alpha Q \\ -\alpha Q & 0 & A^1 & 0 \\ 0 & -\alpha Q & 0 & A^2 \end{bmatrix} \cdot \begin{bmatrix} \mathfrak{u}_{0,1} \\ \mathfrak{u}_{0,2} \\ \mathfrak{u}_1 \\ \mathfrak{u}_2 \end{bmatrix} \tag{17}$$

As in the previous paragraph, let us compute the spectrum of the local multi-trace operator $(1-\alpha)(A - \mathrm{Id}) + \alpha(A - \Pi)$. Here again, it suffices to determine the spectrum of $(A - \alpha\Pi)$. Once again we have $(A - \alpha\Pi)^2 = A^2 + \alpha^2\Pi^2 - \alpha(\Pi A + A\Pi) = (1+\alpha^2)\,\mathrm{Id} - \alpha(\Pi A + A\Pi)$. Besides, taking account of (15), a direct and thorough calculus shows that $Q \cdot A^0_{j,j} \cdot Q = -A^j$. So if we compute explicitly the expression of $\Pi A + A\Pi$ taking account of this identity, we obtain

$$\Pi A + A\Pi = \begin{bmatrix} 0 & 0 & 0 & A^0_{1,2}Q \\ 0 & 0 & A^0_{2,1}Q & 0 \\ 0 & QA^0_{1,2} & 0 & 0 \\ QA^0_{2,1} & 0 & 0 & 0 \end{bmatrix}$$

This time we have $\Pi A + A\Pi \neq 0$. Let us compute the square of this operator. Since $Q^2 = \mathrm{Id}$, we obtain

$$(\Pi A + A\Pi)^2 = \begin{bmatrix} A^0_{1,2}A^0_{2,1} & 0 & 0 & 0 \\ 0 & A^0_{2,1}A^0_{1,2} & 0 & 0 \\ 0 & 0 & QA^0_{1,2}A^0_{2,1}Q & 0 \\ 0 & 0 & 0 & QA^0_{2,1}A^0_{1,2}Q \end{bmatrix}$$

Now let us have a closer look at each of the operators $A^0_{1,2}A^0_{2,1}$ and $A^0_{2,1}A^0_{1,2}$. Observe that $-A^0_{j,k} = 2\,Q \cdot \gamma^j \cdot G^k \cdot Q$ for $j \neq k$. According to the second equality in (7), we have $\gamma^1 \cdot G^2 \cdot \gamma^2 \cdot G^1 = 0$. As a consequence, we obtain

$$\begin{aligned} A^0_{1,2}A^0_{2,1} &= 4\,\big(Q \cdot \gamma^1 \cdot G^2 \cdot Q\big) \cdot \big(Q \cdot \gamma^2 \cdot G^1 \cdot Q\big) \\ &= 4\,Q \cdot (\gamma^1 \cdot G^2 \cdot \gamma^2 \cdot G^1) \cdot Q = 0 \end{aligned}$$

We show in a similar manner that $A^0_{2,1}A^0_{1,2}$. To conclude we have $(\Pi A + A\Pi)^2 = 0$. Such a nilpotent operator has a non-empty spectrum (see [21, Thm.10.13]) that is reduced to $\{0\}$ according to the spectral mapping theorem [21, Thm.10.28], which implies of course that $\mathfrak{S}(\Pi A + A\Pi) = \{0\}$. Finally we obtain the following spectrum, like in the previous paragraph,

$$\mathfrak{S}(A - \Pi_\alpha) \subset \{\; -1 + \alpha + \sqrt{1+\alpha^2}, -1 + \alpha - \sqrt{1+\alpha^2}\;\}.$$

6 Spectrum of the operator in a general configuration

For both examples of the previous two paragraphs, the spectrum of the local multi-trace operator only consisted in the two eigenvalues $-1+\alpha\pm\sqrt{1+\alpha^2}$ in the case where all wave numbers equal. Besides, during the calculations above, the geometry of the interfaces never came into play. In the present section we will show that these are actually general results that hold for any number of sub-domain arranged arbitrarily, provided the geometry does not involve any junction point.

To investigate this question in the general case, we need to introduce further notations. Recall that each boundary can be decomposed in the following manner $\Gamma_j = \cup_{k\neq j}\Gamma_{j,k}$. We will decompose traces accordingly. For a given pair j,k with $j\neq k$ we define

$$\begin{aligned} &\mathrm{R}^j_k : \mathbb{H}(\Gamma_j)\to\mathbb{H}(\Gamma_{j,k})\\ &\mathrm{R}^j_k(\mathfrak{v}) := \mathfrak{v}|_{\Gamma_{j,k}}\end{aligned}$$

To reformulate the above definition fully explicitly, for $\mathfrak{v}=(v,q)\in\mathbb{H}(\Gamma_{j,k}) = \mathrm{H}^{1/2}(\Gamma_{j,k})\times\mathrm{H}^{-1/2}(\Gamma_{j,k})$ and $\mathfrak{u}=(u,p)\in\mathbb{H}(\Gamma_j)=\mathrm{H}^{1/2}(\Gamma_j)\times\mathrm{H}^{-1/2}(\Gamma_j)$, we have $\mathfrak{v}=\mathrm{R}^j_k(\mathfrak{u})$ if and only if $u|_{\Gamma_{j,k}}=v$ and $p|_{\Gamma_{j,k}}=q$. The adjoint of these restriction operators are given by

$$\begin{aligned} &(\mathrm{R}^j_k)^* : \mathbb{H}(\Gamma_{j,k})\to\mathbb{H}(\Gamma_j)\\ &(\mathrm{R}^j_k)^*(\mathfrak{v}) := \mathfrak{v}\cdot 1_{\Gamma_{j,k}}\end{aligned}$$

Decomposing traces with the embedding/restriction operators that we have just defined, each $\mathrm{A}^j:\mathbb{H}(\Gamma_j)\to\mathbb{H}(\Gamma_j)$ induces a matrix of integral operators denoted $[\mathrm{A}^j]$ with maximal size $(n-1)\times(n-1)$ given by

$$[\mathrm{A}^j] := \begin{bmatrix} \mathrm{A}^j_{0,0} & \cdots & \mathrm{A}^j_{0,n}\\ \vdots & & \vdots\\ \mathrm{A}^j_{n,0} & \cdots & \mathrm{A}^j_{n,n}\end{bmatrix} \qquad \mathrm{A}^j_{k,m} := \mathrm{R}^j_k\cdot\mathrm{A}^j\cdot(\mathrm{R}^j_m)^* \tag{18}$$

In the notation above, it should be understood that the rows and the columns associated to indices k such that $\Gamma_{j,k}=\partial\Omega_j\cap\partial\Omega_k=\emptyset$ must be omitted. The row/column associated to $k=j$ is to be omitted as well. For each sub-domain Ω_j, we will need to consider the set of indices

$$\mathscr{I}_j := \{\ k \mid k\neq j,\ \partial\Omega_j\cap\partial\Omega_k\neq\emptyset\}.$$

Hence the matrix in (18) is square with $\mathrm{card}(\mathscr{I}_j)$ rows. If we plug the definition of the restriction/embedding operators R^j_k into the definition (9) of the operator A^j and the potential operator (6), we obtain an explicit formula for $\mathrm{A}^j_{k,m}$ in the case where $k\neq m$ namely,

$$\begin{aligned} -\Big[\!\Big[\ \mathrm{A}^j_{k,m}\Big(\begin{array}{c}u\\p\end{array}\Big),\Big(\begin{array}{c}v\\q\end{array}\Big)\ \Big]\!\Big]_{\Gamma_{j,k}} = &\\ \int_{\Gamma_{j,k}}\int_{\Gamma_{j,m}} \mathrm{G}_\kappa(\boldsymbol{x}-\boldsymbol{y})\big[\ &(\mathbf{n}(\boldsymbol{x})\times\nabla u(\boldsymbol{x}))\cdot(\mathbf{n}(\boldsymbol{y})\times\nabla v(\boldsymbol{y}))\\ &-\kappa^2\mathbf{n}(\boldsymbol{x})\cdot\mathbf{n}(\boldsymbol{y})u(\boldsymbol{x})v(\boldsymbol{y}) - p(\boldsymbol{x})q(\boldsymbol{y})\ \big]\,d\sigma(\boldsymbol{y})d\sigma(\boldsymbol{x})\\ +\int_{\Gamma_{j,k}}\int_{\Gamma_{j,m}} (\nabla\mathrm{G}_\kappa)(\boldsymbol{x}-\boldsymbol{y})\cdot\big[\ &q(\boldsymbol{y})u(\boldsymbol{x})\mathbf{n}(\boldsymbol{x}) - p(\boldsymbol{x})v(\boldsymbol{y})\mathbf{n}(\boldsymbol{y})\ \big]\,d\sigma(\boldsymbol{y})d\sigma(\boldsymbol{x})\end{aligned} \tag{19}$$

for any $(u,p) \in \mathbb{H}(\Gamma_{j,m})$ and $(v,q) \in \mathbb{H}(\Gamma_{j,k})$. Note that the expression above does not hold for $k = m$. Expression (19) clearly shows that the operators $\mathrm{A}^j_{k,m}$ are compact for $k \neq m$ since it only involves smooth kernels. The following lemma yields several remarkable identities satisfied by the elements of the decomposition (18).

Lemma 6.1.
For any $j = 0 \dots n$, and any $k, l, m \in \mathscr{I}_j$ with $k \neq m$ and $k \neq l$ we have

i) $(\mathrm{A}^j_{k,k})^2 = \mathrm{Id}$,

ii) $\mathrm{Q} \cdot \mathrm{A}^j_{k,k} = -\mathrm{A}^k_{j,j} \cdot \mathrm{Q}$ *if* $\kappa_j = \kappa_k$

iii) $\mathrm{A}^j_{l,k} \cdot \mathrm{A}^j_{k,m} = 0$,

Proof:

Pick an arbitrary $j = 0 \dots n$ that will be fixed until the end of the proof. Take a arbitrary $k, m \in \mathscr{I}_j$ with $k \neq m$. Let $\mathcal{O} \subset \mathbb{R}^d$ be the maximal open set satisfying $\partial\mathcal{O} = \Gamma_{j,m}$ and $\Omega_j \cap \mathcal{O} = \emptyset$. Take an arbitrary $\mathfrak{v} \in \mathbb{H}(\Gamma_{j,m})$, and denote $\tilde{\mathfrak{v}} := (\mathrm{R}^j_m)^*\mathfrak{v}$. Observe that $\mathrm{G}^j_\kappa(\tilde{\mathfrak{v}}) \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \mathcal{O})$. Since $\Gamma_{j,k} \cap \Gamma_{j,m} = \emptyset$, this implies in particular that $\mathrm{G}^j_\kappa(\tilde{\mathfrak{v}})$ does not admit any jump across $\Gamma_{j,k}$. As a consequence, we have

$$\mathrm{A}^j_{k,m}(\mathfrak{v}) = 2\,\mathrm{R}^j_k \cdot \gamma^j \cdot \mathrm{G}^j(\tilde{\mathfrak{v}}) = 2\,\mathrm{R}^j_k \cdot \gamma^j_c \cdot \mathrm{G}^j(\tilde{\mathfrak{v}})$$

Now observe that, if $\mathfrak{w} \in \mathcal{C}^{\mathrm{out}}_\kappa(\Omega_j)$, then $(\mathrm{R}^j_k)^*\mathrm{R}^j_k(\mathfrak{w}) \in \mathcal{C}^{\mathrm{out}}_\kappa(\Omega_j)$ for any $k \in \mathscr{I}_j$. Taking $\mathfrak{w} = \gamma^j_c \cdot \mathrm{G}^j(\tilde{\mathfrak{v}})$, we see that $\tilde{\mathfrak{w}} = (\mathrm{R}^j_k)^*\mathrm{A}^j_{k,m}(\mathfrak{v}) \in \mathcal{C}^{\mathrm{out}}_\kappa(\Omega_j)$. According to the integral representation Theorem 3.1, this implies $\mathrm{G}^j_\kappa(\tilde{\mathfrak{w}})(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega_j$. In particular we have

$$2\,\mathrm{R}^j_l \gamma^j \mathrm{G}^j_\kappa(\tilde{\mathfrak{w}}) = \mathrm{A}^j_{l,k} \cdot \mathrm{A}^j_{k,m}(\mathfrak{v}) = 0.$$

This establishes *iii)*. Now we know that $(\mathrm{A}^j)^2 = \mathrm{Id}$. This implies that, for any $k \in \mathscr{I}_j$, we have $\sum_{m \in \mathscr{I}_j} \mathrm{A}^j_{k,m} \cdot \mathrm{A}^j_{m,k} = \mathrm{Id}$. But according to *iii)*, all the terms of this sum vanish, except for $k = m$ which establishes *i)*.

To prove *ii)*, observe that $\mathrm{Q} \cdot \mathrm{R}^j_k \cdot \{\gamma^j\} = \mathrm{R}^k_j \cdot \{\gamma^k\}$. On the other hand, the explicit expression of potential operators given by (6) shows that $\mathrm{G}^j_\kappa \cdot (\mathrm{R}^j_k)^* = -\mathrm{G}^k_\kappa \cdot (\mathrm{R}^k_j)^* \cdot \mathrm{Q}$ in the case where $\kappa_j = \kappa_k$. Combining these two identities we obtain $\mathrm{Q} \cdot \mathrm{A}^j_{k,k} = 2\,\mathrm{Q} \cdot \mathrm{R}^j_k \cdot \{\gamma^j\} \cdot \mathrm{G}^j_\kappa \cdot (\mathrm{R}^j_k)^* = -2\,\mathrm{R}^k_j \cdot \{\gamma^k\} \cdot \mathrm{G}^k_\kappa \cdot (\mathrm{R}^k_j)^* \cdot \mathrm{Q} = -\mathrm{A}^k_{j,j} \cdot \mathrm{Q}$.

□

Next we need to introduce an operator involving only the diagonal blocks of the matrix representing A in the decomposition (18), without any term coupling different interfaces. Define $\mathrm{D} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ by the explicit formula

$$[\![\mathrm{D}(\mathfrak{u}), \mathfrak{v}]\!] := \sum_{j=0}^{n} \sum_{k \in \mathscr{I}_j} [\![\, \mathrm{A}^j_{k,k}(\mathfrak{u}_{j,k}), \mathfrak{v}_{j,k} \,]\!]_{\Gamma_{j,k}} \qquad \forall \mathfrak{u}, \mathfrak{v} \in \mathbb{H}(\Sigma) \tag{20}$$

$$\text{where} \quad \mathfrak{u}_{j,k} := \mathrm{R}^j_k(\mathfrak{u}_j), \quad \mathfrak{v}_{j,k} := \mathrm{R}^j_k(\mathfrak{v}_j)$$

Lemma 6.2.
We have $(\mathrm{D})^2 = \mathrm{Id}$*. Moreover, if all wave numbers equal i.e.* $\kappa_0 = \kappa_1 = \cdots = \kappa_n$*, we have* $\Pi \cdot \mathrm{D} + \mathrm{D} \cdot \Pi = 0$*.*

Proof:

According to Lemma 6.1, we already know that $(\mathrm{A}^j_{k,k})^2 = \mathrm{Id}$. Pick an arbitrary pair of traces $\mathfrak{u}, \mathfrak{v} \in \mathbb{H}(\Sigma)$, and set $\mathfrak{u}_{j,k} := \mathrm{R}^j_k(\mathfrak{u}_j)$ and $\mathfrak{v}_{j,k} := \mathrm{R}^j_k(\mathfrak{v}_j)$ for all j, and all $k \in \mathscr{I}_j$. Applying Formula (20) twice yields

$$
\begin{aligned}
[\![\mathrm{D}^2(\mathfrak{u}), \mathfrak{v}]\!] &:= \sum_{j=0}^{n} \sum_{k\in\mathscr{I}_j} [\![(\mathrm{A}^j_{k,k})^2 \mathfrak{u}_{j,k}, \mathfrak{v}_{j,k}]\!]_{\Gamma_{j,k}} \\
&= \sum_{j=0}^{n} \sum_{k\in\mathscr{I}_j} [\![\mathfrak{u}_{j,k}, \mathfrak{v}_{j,k}]\!]_{\Gamma_{j,k}} = \sum_{j=0}^{n} [\![\mathfrak{u}_j, \mathfrak{v}_j]\!]_{\Gamma_j} = [\![\mathfrak{u}, \mathfrak{v}]\!].
\end{aligned}
$$

which establishes that $\mathrm{D}^2 = \mathrm{Id}$. To establish the second statement, let us first point out that the definition (10) of the operator Π can be rewritten $[\![\Pi(\mathfrak{u}), \mathfrak{v}]\!] = \sum_{j=0}^{n} \sum_{k\in\mathscr{I}_j} [\![\mathrm{Q}(\mathfrak{u}_{k,j}), \mathfrak{v}_{j,k}]\!]_{\Gamma_{j,k}}$. Combining this with the definition of the operator D, and using Property *iii)* of Lemma 6.1, we obtain

$$
\begin{aligned}
[\![\Pi \cdot \mathrm{D}(\mathfrak{u}), \mathfrak{v}]\!] &:= \sum_{j=0}^{n} \sum_{k\in\mathscr{I}_j} [\![\mathrm{Q}\, \mathrm{A}^k_{j,j} \mathfrak{u}_{k,j}, \mathfrak{v}_{j,k}]\!]_{\Gamma_{j,k}} \\
&:= \sum_{j=0}^{n} \sum_{k\in\mathscr{I}_j} -[\![\mathrm{A}^j_{k,k}\, \mathrm{Q}(\mathfrak{u}_{k,j}), \mathfrak{v}_{j,k}]\!]_{\Gamma_{j,k}} = -[\![\mathrm{D} \cdot \Pi(\mathfrak{u}), \mathfrak{v}]\!]
\end{aligned}
$$

□

Lemma 6.3.
The operator $\mathrm{T} := \mathrm{A} - \mathrm{D}$ *is compact and satisfies* $\mathrm{T}^2 = 0$*.*

Proof:

The operator T only involves terms $\mathrm{A}^j_{k,m}$ with $k \neq m$. Since $\Gamma_{j,k} \cap \Gamma_{j,m} = \emptyset$, these operators defined by (19) only involve smooth kernels. So each $\mathrm{A}^j_{k,m}, k \neq m$ is compact, and T is compact itself. Let us compute explicitly the expression of T^2. Pick arbitrary $\mathfrak{u}, \mathfrak{v} \in \mathbb{H}(\Sigma)$, and set $\mathfrak{u}_{j,k} := \mathrm{R}^j_k(\mathfrak{u}_j)$ and $\mathfrak{v}_{j,k} := \mathrm{R}^j_k(\mathfrak{v}_j)$ for all j, and all $k \in \mathscr{I}_j$. We have

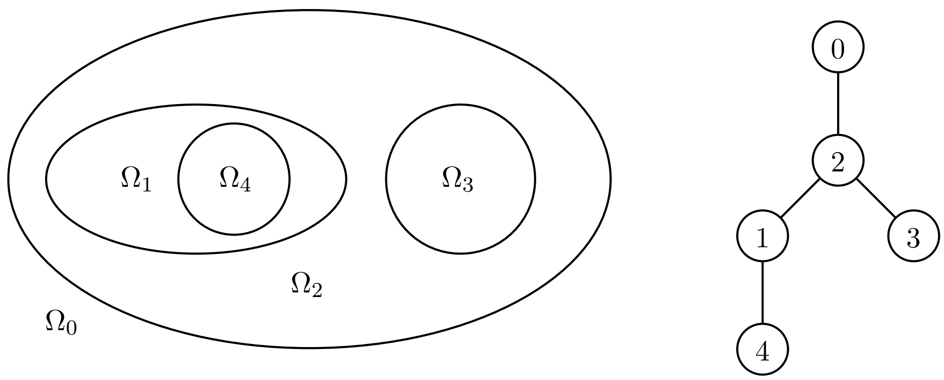
$$
\begin{aligned}
[\![\mathrm{T}(\mathfrak{u}), \mathfrak{v}]\!] &:= \sum_{j=0}^{n} \sum_{\substack{k,m\in\mathscr{I}_j \\ k\neq m}} [\![\mathrm{A}^j_{k,m}(\mathfrak{u}_{j,m}), \mathfrak{v}_{j,k}]\!]_{\Gamma_{j,k}} \\
[\![\mathrm{T}^2(\mathfrak{u}), \mathfrak{v}]\!] &:= \sum_{j=0}^{n} \sum_{\substack{k,l,m\in\mathscr{I}_j \\ k\neq l, l\neq m}} [\![\mathrm{A}^j_{k,l}\mathrm{A}^j_{l,m}(\mathfrak{u}_{j,m}), \mathfrak{v}_{j,k}]\!]_{\Gamma_{j,k}}
\end{aligned}
$$

To conclude it remains to apply property *ii)* of Lemma 6.1. Since, for each term $\mathrm{A}^j_{k,l}\mathrm{A}^j_{l,m}$ we have $k \neq l$ and $m \neq l$, the whole sum vanishes, and we have $[\![\mathrm{T}^2(\mathfrak{u}), \mathfrak{v}]\!] = 0$. □

Note that it is a direct consequence of the above lemma and Proposition 3.3 that D also satisfies a Garding inequality. In addition, since $\Pi^2 = \mathrm{Id}$, a recurrence argument combined with the previous lemma readily leads to the following corollary. Note that, remarkably, this result holds *without* any assumption on the wave numbers.

Corollary 6.1.
We have $(\Pi\mathrm{T} + \mathrm{T}\Pi)^k = (\Pi\mathrm{T})^k + (\mathrm{T}\Pi)^k, \quad \forall k \geq 0.$

Now let us formulate a few elementary and useful remarks concerning the geometrical arrangement of the interfaces. Let $\Upsilon = \{0, 1, \dots, n\}$, and say that two indices j, k are adjacent if $\partial\Omega_j \cap \partial\Omega_k \neq \emptyset$. This adjacency relation endow Υ with a graph structure. We order the elements of Υ, writing $j \prec k$ if Ω_j is included in a bounded connected component of $\mathbb{R}^d \setminus \overline{\Omega}_k$. This induces an tree struture on Υ. In particular Υ is a tree and does not admit any (simple cycle), see for example [1, Chap.16]. And it does not admit chain with a length larger than n. The picture below provides an example of such a tree structure.



Proposition 6.1.
In the case where all wave numbers equal $\kappa_0 = \kappa_1 = \cdots = \kappa_n$, *we have* $(\Pi\mathrm{T})^n = (\mathrm{T}\Pi)^n = 0$ *where* n *is the number of sub-domains.*

Proof:
First of all note that $(\mathrm{T}\Pi)^n = \Pi(\Pi\mathrm{T})^n\Pi$, so we only need to prove the result for $(\Pi\mathrm{T})^n$. We start by simply writing down the explicit expression of the operator $(\Pi\mathrm{T})^n$. For any $\mathfrak{u}, \mathfrak{v} \in \mathbb{H}(\Sigma)$, setting $\mathfrak{u}_{j,k} := \mathrm{R}^j_k(\mathfrak{u}_j)$ and $\mathfrak{v}_{j,k} := \mathrm{R}^j_k(\mathfrak{v}_j)$, we have

$$\begin{aligned}
\llbracket \Pi\mathrm{T}(\mathfrak{u}), \mathfrak{v} \rrbracket \quad &:= \sum_{i_0=0}^{n} \sum_{i_1 \in \mathscr{I}_{i_0}} \sum_{\substack{i_2 \in \mathscr{I}_{i_1} \\ i_2 \neq i_0}} \llbracket \mathrm{QA}^{i_1}_{i_0,i_2}(\mathfrak{u}_{i_1,i_2}), \mathfrak{v}_{i_0,i_1} \rrbracket_{\Gamma_{i_0,i_1}} \\
\llbracket (\Pi\mathrm{T})^2(\mathfrak{u}), \mathfrak{v} \rrbracket \quad &:= \sum_{i_0=0}^{n} \sum_{i_1 \in \mathscr{I}_{i_0}} \sum_{\substack{i_2 \in \mathscr{I}_{i_1} \\ i_2 \neq i_0}} \sum_{\substack{i_3 \in \mathscr{I}_{i_2} \\ i_3 \neq i_1}} \llbracket \mathrm{QA}^{i_1}_{i_0,i_2} \mathrm{QA}^{i_2}_{i_1,i_3}(\mathfrak{u}_{i_2,i_3}), \mathfrak{v}_{i_0,i_1} \rrbracket_{\Gamma_{i_0,i_1}}
\end{aligned}$$

In the expression above we have $i_2 \neq i_0$ and $i_3 \neq i_1$ due to the very definition of the operator T. Applying recursively the formulas derived above for $\Pi\mathrm{T}$ finally leads to the following explicit expression for $(\Pi\mathrm{T})^n$,

$$\llbracket (\Pi\mathrm{T})^n(\mathfrak{u}),\mathfrak{v}\rrbracket \;:= \sum_{i_0=0}^{n}\sum_{i_1\in\mathscr{I}_{i_0}}\sum_{\substack{i_2\in\mathscr{I}_{i_1}\\ i_2\neq i_0}}\sum_{\substack{i_3\in\mathscr{I}_{i_2}\\ i_3\neq i_1}}\sum_{\substack{i_4\in\mathscr{I}_{i_3}\\ i_3\neq i_2}}\cdots\cdots\sum_{\substack{i_{n+1}\in\mathscr{I}_{i_n}\\ i_{n+1}\neq i_{n-1}}} \llbracket \mathrm{QA}^{i_1}_{i_0,i_2}\mathrm{QA}^{i_2}_{i_1,i_3}\cdots\mathrm{QA}^{i_n}_{i_{n-1},i_{n+1}}(\mathfrak{u}_{i_n,i_{n+1}}),\mathfrak{v}_{i_0,i_1}\rrbracket_{\Gamma_{i_0,i_1}} \tag{21}$$

The sum in the expression above is taken over all the sequence of indices $i_0, i_1, \ldots, i_{n+1}$ satisfying the constraints $i_k \in \mathscr{I}_{i_{k-1}}$ (which implies in particular that $i_k \neq i_{k-1}$) and $i_{k+1} \neq i_{k-1}$ for all $k = 1 \ldots n$. Each i_k is the index of the sub-domain Ω_{i_k}, and $\Omega_{i_{k+1}}$ is adjacent to Ω_{i_k} since $i_{k+1} \in \mathscr{I}_{i_k}$, hence those sequences $i_0, i_1, \ldots, i_{n+1}$ are actually chains of length exactly $n+1$ of the tree Υ. But since Υ only admits n elements and is a tree, it does not contain any such chain. This implies that the summation in (21) is taken over an empty set. Hence $\llbracket (\Pi\mathrm{T})^n(\mathfrak{u}),\mathfrak{v}\rrbracket = 0$, and since $\mathfrak{u},\mathfrak{v}$ were chosen arbitrarily, this finally implies $(\Pi\mathrm{T})^n = 0$.
□

The spectrum of the local multi-trace operator can now easily be deduced from what precedes, in the case where all wave numbers equal. Note that the next result states equality and not just inclusion.

Theorem 6.1.
Assume that all wave numbers are equal i.e. $\kappa_0 = \kappa_1 = \cdots = \kappa_n$. Let $\mathfrak{S}_p(\mathrm{A}-\alpha\Pi)$ refer to the point spectrum of $\mathrm{A}-\alpha\Pi$ i.e. the set of its eigenvalues. Then the spectrum of this operators coincides with the point spectrum, and it is given by

$$\mathfrak{S}(\mathrm{A}-\alpha\Pi) = \mathfrak{S}_p(\mathrm{A}-\alpha\Pi) = \{+\sqrt{1+\alpha^2}, -\sqrt{1+\alpha^2}\}.$$

Proof:

The result is clear if $\alpha = 0$ since $(\mathrm{Id}+\mathrm{A})/2$ is a projector, so for the remaining of this proof, we will assume that $\alpha \neq 0$. Taking the square of the above operator, and using Lemma 6.2, yields $(\mathrm{A}-\alpha\Pi)^2 = (1+\alpha^2)\mathrm{Id} - \alpha(\mathrm{T}\Pi + \Pi\mathrm{T})$. Then it is a direct consequence of Corollary 6.1 and Proposition 6.1 that the operator $\mathrm{T}\Pi + \Pi\mathrm{T}$ is nilpotent. Hence according to [21, Thm.10.13] and the spectral mapping theorem [21, Thm.10.28], we have $\mathfrak{S}(\mathrm{T}\Pi + \Pi\mathrm{T}) = \{0\}$. This also shows that $\mathfrak{S}((\mathrm{A}-\alpha\Pi)^2) = \{1+\alpha^2\}$, hence applying once again the spectral mapping theorem, we finally conclude that $\mathfrak{S}(\mathrm{A}-\alpha\Pi) \subset \{+\sqrt{1+\alpha^2}, -\sqrt{1+\alpha^2}\}$.

Denote for a moment $f(\lambda) := \lambda^2$. Then clearly $\mathfrak{S}_p(\mathrm{T}\Pi + \Pi\mathrm{T}) = \{0\}$ since $\mathrm{T}\Pi + \Pi\mathrm{T}$ is nilpotent. Moreover, according to the "point spectrum counterpart" of the spectral mapping theorem [21, Thm.10.33], we have $f(\mathfrak{S}_p(\mathrm{A}-\alpha\Pi)) = \mathfrak{S}_p(f(\mathrm{A}-\alpha\Pi)) = \{1+\alpha^2\}$. Hence we conclude also that $\mathfrak{S}_p(\mathrm{A}-\alpha\Pi) \subset \{+\sqrt{1+\alpha^2}, -\sqrt{1+\alpha^2}\}$. If we can prove that the previous inclusion is actually an equality, then the proof will be finished. It suffices to show that, if $\lambda \in \mathfrak{S}_p(\mathrm{A}-\alpha\Pi)$, then we also have $-\lambda \in \mathfrak{S}_p(\mathrm{A}-\alpha\Pi)$.

Take an eigenvector $\mathfrak{u} \in \mathbb{H}(\Sigma) \setminus \{0\}$ of $\mathrm{A}-\alpha\Pi$ associated to the eigenvalue λ. Since $\mathrm{A}^2 = \Pi^2 = \mathrm{Id}$, we have $\mathrm{A}(\mathrm{A}\Pi - \Pi\mathrm{A}) = -(\mathrm{A}\Pi - \Pi\mathrm{A})\mathrm{A}$ and $\Pi(\mathrm{A}\Pi - \Pi\mathrm{A}) = -(\mathrm{A}\Pi - \Pi\mathrm{A})\Pi$. As a consequence $(\mathrm{A}-\alpha\Pi)(\mathrm{A}\Pi - \Pi\mathrm{A})\mathfrak{u} = -(\mathrm{A}\Pi - \Pi\mathrm{A})(\mathrm{A}-\alpha\Pi)\mathfrak{u} = -\lambda(\mathrm{A}\Pi - \Pi\mathrm{A})\mathfrak{u}$. Hence, $-\lambda$ is an eigenvalue of $\mathrm{A}-\alpha\Pi$ if λ is, provided that $(\mathrm{A}\Pi - \Pi\mathrm{A})\mathfrak{u} \neq 0$. Observe that, since $(\mathrm{A}-\alpha\Pi)\mathfrak{u} = \lambda\mathfrak{u}$ and $\mathrm{A}^2 = \Pi^2 = \mathrm{Id}$, we have $\Pi\mathrm{A}\mathfrak{u} = \alpha\mathfrak{u} + \lambda\Pi\mathfrak{u}$ and $\mathrm{A}\Pi\mathfrak{u} = (1/\alpha)\mathfrak{u} - (\lambda/\alpha)\mathrm{A}\mathfrak{u}$.

Summing these two identities yields

$$(\Pi \mathrm{A} - \mathrm{A}\Pi)\mathfrak{u} = (\alpha - 1/\alpha)\mathfrak{u} + \frac{\lambda}{\alpha}(\mathrm{A} + \alpha\Pi)\mathfrak{u}.$$

If we can prove that 0 is not an eigenvalue of $(\alpha - 1/\alpha)\mathrm{Id} + \frac{\lambda}{\alpha}(\mathrm{A} + \alpha\Pi)$, this will show that $(\Pi\mathrm{A} - \mathrm{A}\Pi)\mathfrak{u} \neq 0$. From the first part of the proof, we know that the spectrum of $\mathrm{A} + \alpha\Pi$ is included in $\{+\sqrt{1+\alpha^2}, -\sqrt{1+\alpha^2}\}$. Besides λ equals $+\sqrt{1+\alpha^2}$ or $-\sqrt{1+\alpha^2}$. As a consequence the spectrum of $(\alpha - 1/\alpha)\mathrm{Id} + \frac{\lambda}{\alpha}(\mathrm{A} + \alpha\Pi)$ only contains the values

$$\alpha - \frac{1}{\alpha} \pm \frac{\lambda}{\alpha}\sqrt{1+\alpha^2} = \frac{\alpha^2 - 1 \pm (1+\alpha^2)}{\alpha} = \;\; 2\alpha \;\text{ or }\; -2/\alpha$$

Since $\alpha \in \mathbb{C} \setminus \{0\}$, we have $2\alpha \neq 0$ and $2/\alpha \neq 0$. As a consequence the spectrum of $(\alpha - 1/\alpha)\mathrm{Id} + \frac{\lambda}{\alpha}(\mathrm{A} + \alpha\Pi)$ does not contain 0 so $(\Pi\mathrm{A} - \mathrm{A}\Pi)\mathfrak{u} \neq 0$ necessarily. □

The previous theorem can be reformulated as follows.

Corollary 6.2.
Assume that all wave numbers are equal $\kappa_0 = \kappa_1 = \cdots = \kappa_n$. Then for any pair of complex numbers $\alpha, \beta \in \mathbb{C}$ the operator $\mathrm{A} + \alpha\Pi + \beta\mathrm{Id}$ is invertible if and only if $\beta^2 - \alpha^2 \neq 1$.

Theorem 6.1 also leads directly to explicit expression for the spectrum of the local multi-trace operator. Thanks to Fredholm theory, this implies a well-posedness result.

Corollary 6.3.
For any $\alpha \in \mathbb{C} \setminus \{0\}$, the operator $\mathrm{L}_\alpha := (1-\alpha)(\mathrm{A} - \mathrm{Id}) + \alpha(\mathrm{A} - \Pi)$ is invertible. Moreover, in the case where all wave numbers are equal $\kappa_0 = \kappa_1 = \cdots = \kappa_n$, its spectrum equals its point spectrum and $\mathfrak{S}(\mathrm{L}_\alpha) = \mathfrak{S}_p(\mathrm{L}_\alpha) = \{-1 + \alpha - \sqrt{1+\alpha^2}, -1 + \alpha + \sqrt{1+\alpha^2}\}$.

Proof:

Assume that the wave numbers $\kappa_0, \kappa_1, \ldots, \kappa_n$ are arbitrary elements of $(0, +\infty)$, and consider any $\alpha \in \mathbb{C} \setminus \{0\}$. Let $\mathrm{A}_\star$ refer to the operator defined in the same manner as A but with wave numbers all equal to $\kappa_\star = \imath$. Then the operator $\mathrm{A} - \mathrm{A}_\star : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ is compact as it only involves integral operators with regular kernels, see [22, Lemma 3.9.8]. Then Theorem 6.1 shows that $(1-\alpha)(\mathrm{A}_\star - \mathrm{Id}) + \alpha(\mathrm{A}_\star - \Pi)$ is invertible, since its eigenvalues $-1 + \alpha \pm \sqrt{1+\alpha^2}$ differ from 0 as $\alpha \neq 0$. Hence $(1-\alpha)(\mathrm{A} - \mathrm{Id}) + \alpha(\mathrm{A} - \Pi)$ is a compact perturbation of an isomorphism. According to Fredholm-Riesz-Schauder theory (see [16, Chap.2] for example), this operator is invertible if and only if it is one-to-one. Since it is injective according to Proposition 4.1, we finally conclude that $(1-\alpha)(\mathrm{A} - \mathrm{Id}) + \alpha(\mathrm{A} - \Pi)$ is an isomorphism. The second statement above concerning the spectrum is a trivial consequence of Theorem 6.1. □

In the case where wave numbers take arbitrary values the spectrum is not reduced to $-1 + \alpha \pm \sqrt{1+\alpha^2}$ anymore. However a difference of wave numbers only induces compact perturbation of integral operators so that, in the general case, this result still indicates the location of accumulation points of the spectrum.

Corollary 6.4.
For any $\alpha \in \mathbb{C} \setminus \{0\}$, set $\mathrm{L}_\alpha := (1-\alpha)(\mathrm{A} - \mathrm{Id}) + \alpha(\mathrm{A} - \Pi)$. Then any element $\lambda \in \mathfrak{S}(\mathrm{L}_\alpha) \setminus \{-1 + \alpha + \sqrt{1+\alpha^2}, -1 + \alpha - \sqrt{1+\alpha^2}\}$ is an isolated eigenvalue with $\dim(\ker(\mathrm{L}_\alpha - \lambda\mathrm{Id})) < +\infty$. Moreover the two values $-1 + \alpha \pm \sqrt{1+\alpha^2}$ are the only possible accumulation points of $\mathfrak{S}(\mathrm{L}_\alpha)$.

Proof:

Denote $\mu_\alpha^\pm := -1+\alpha\pm\sqrt{1+\alpha^2}$, and set $\mathcal{L}(\lambda) := \mathrm{L}_\alpha - \lambda\mathrm{Id}$. Then $\lambda\mapsto\mathcal{L}(\lambda)$ is an analytic operator pencil, and it is Fredholm valued for $\lambda\neq\mu_\alpha^\pm$. Indeed take any $\lambda\in\mathbb{C}\setminus\{\mu_\alpha^+,\mu_\alpha^-\}$, and define the operator L'_α in the same manner as L_α except that all wave numbers are taken equal to κ_0. The operator $\mathcal{L}'(\lambda) := \mathrm{L}'_\alpha - \lambda\mathrm{Id}$ is invertible according to Corollary 6.3, and $\mathcal{L}'(\lambda)-\mathcal{L}(\lambda)$ is compact. As a consequence, $\mathcal{L}(\lambda)$ is a compact perturbation of an invertible operator, so it is Fredholm with index 0 and admits finite dimensional kernel.

Further, since L_α is a bounded operator, $\mathcal{L}(\lambda)$ is invertible for $\lambda > \|\mathrm{L}_\alpha\|$, where $\|\;\|$ refers here to the norm naturally associated to continuous operators mapping $\mathbb{H}(\Sigma)$ into itself. As a consequence, we can apply Fredholm analytic theorem (see Appendix A in [12]) which shows that $\mathcal{L}(\lambda)$ is invertible in $\mathbb{C}\setminus\{\mu_\alpha^+,\mu_\alpha^-\}$ except for a countable set of isolated values. Moreover, we have just seen that all these values can only lie in the disc of center 0 and radius $\|\mathrm{L}_\alpha\|$. $\square$

7 Numerical evidences

In this section, we present a series of numerical results confirming the conclusions presented previously. We consider 2-D scattering problems of the form (2) involving three domains. As regards discretization, we consider a uniform paneling $\Sigma^h\simeq\Sigma$ which induces a mesh for each of the sub-domains $\Gamma_j^h\simeq\Gamma_j$, $\Gamma_j^h\subset\Sigma^h$. The discrete spaces $\mathbb{H}_h(\Sigma)$ are constructed on these meshes by means of $\mathbb{P}_1$-Lagrange shape functions for both Dirichlet and Neumann traces

$$\begin{aligned}\mathbb{H}_h(\Sigma) = \{\; & (u_j^h, p_j^h)_{j=0,1,2}\text{ such that}\\ & \forall j=0,1,2,\text{for all panel } e\subset\Gamma_j^h,\;\; u_j^h|_e, p_j^h|_e\in\mathbb{P}_1(e)\;\}.\end{aligned}\tag{22}$$

Denote B_h the matrix associated to the Galerkin discretization of the local multi-trace formulation (12) by means of the discrete space (22), and let us denote M_h the matrix obtained by Galerkin discretization of the bilinear form $(\mathfrak{u},\mathfrak{v})\mapsto[\![\mathfrak{u},\mathfrak{v}]\!]$. We shall focus our attention on the spectrum of the matrix $(\mathrm{M}_h)^{-1}\mathrm{B}_h$ that may be considered as an approximation of the continuous operator associated to Formulation (12).

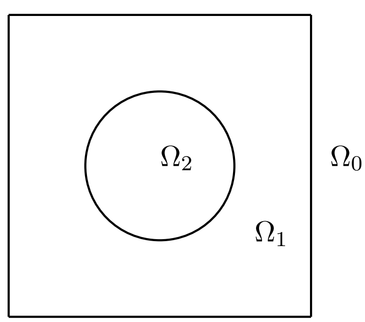


Figure 1: First geometrical configuration

All computations have been achieved on a laptop equipped with a 2-core Intel i7-3520M processor at 2.9GHz with 4 GB of RAM. Meshes have been generated using Gmsh [9] (see also the website `http://geuz.org/gmsh/`). For computation of eigenvalues we relied on the Arpack++ library (see `http://www.ime.unicamp.br/~chico/arpack++/`).

Figure 1 represents a first geometrical configuration. The boundary of Ω_0 is a unit square centered at 0, and the boundary of Ω_2 is a circle of radius 0.5 centered at 0. Figure 2 represents the spectrum of $(\mathrm{M}_h)^{-1}\mathrm{B}_h$ for a mesh width $h = 0.05$ and $\kappa_0 = \kappa_1 = \kappa_2 = 1$.

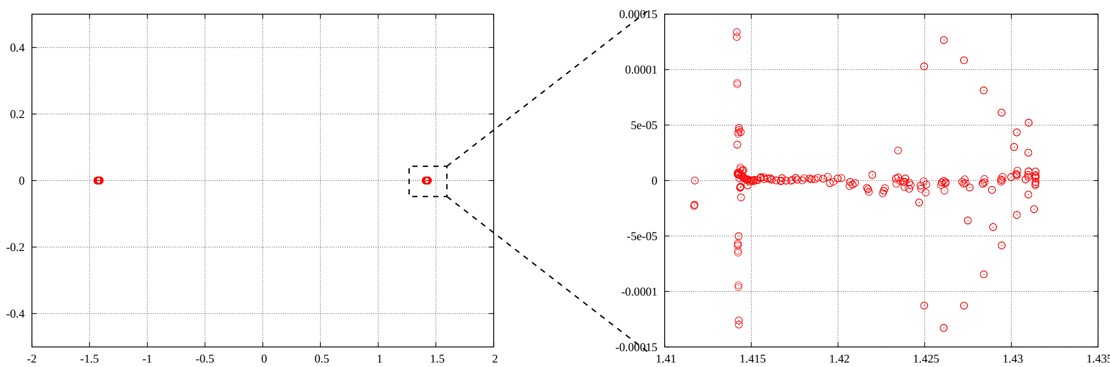


Figure 2: Two clusters of eigenvalues in the case of no contrast of wave numbers (left) and a zoom on the cluster of positive eigenvalues (right). We took $\alpha = 1$ and $\kappa_0 = \kappa_1 = \kappa_2 = 1$.

The spectrum clearly takes the form of two clusters centered at the values $\pm\sqrt{2} = -1 + \alpha \pm \sqrt{1+\alpha^2}$ for $\alpha = 1$. Figure 3 shows the spectrum of the same matrix, with the same geometrical configuration, but with $\alpha = 0.5$ and $\alpha = -0.25$. The formula $-1 + \alpha \pm \sqrt{1+\alpha^2}$ yields the values 0.61803 and -1.6180 for $\alpha = 0.5$ (up to 5 digits), and -0.21922 and -2.2808 for $\alpha = -0.25$, which is consistent with our theory.

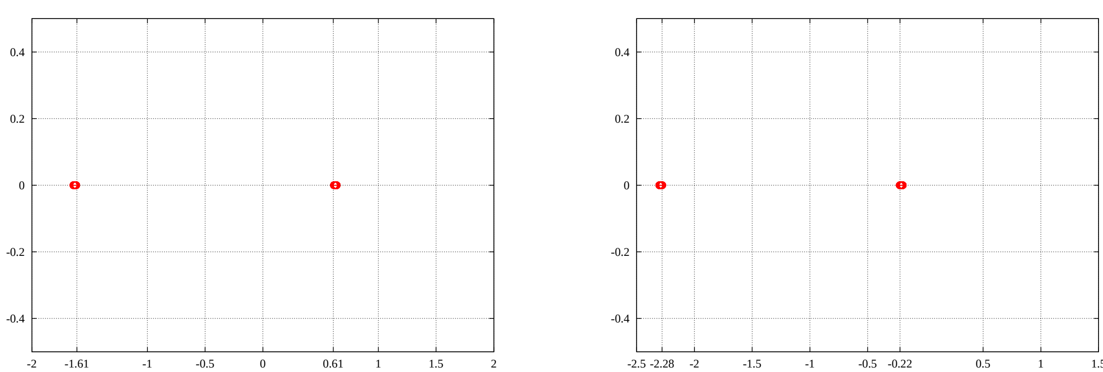


Figure 3: Spectrum of the local multi-trace operator with $\kappa_0 = \kappa_1 = \kappa_2 = 1$ and $\alpha = 0.5$ (left) or $\alpha = -0.25$ (right)

Next, in Figure 4, we consider the case $\alpha = 1$, but wave numbers differ taking the values $\kappa_0 = 1$, $\kappa_1 = 5$ and $\kappa_2 = 2$. The mesh width remains $h = 0.05$. Although the eigenvalues are not clustered anymore, they are more densely grouped around $\pm\sqrt{2}$ suggesting that these are the only two accumulation points of the spectrum of the continuous operator.

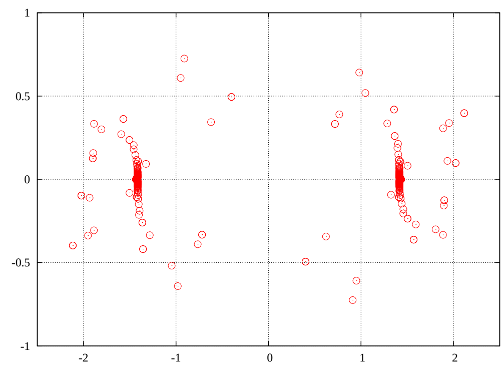
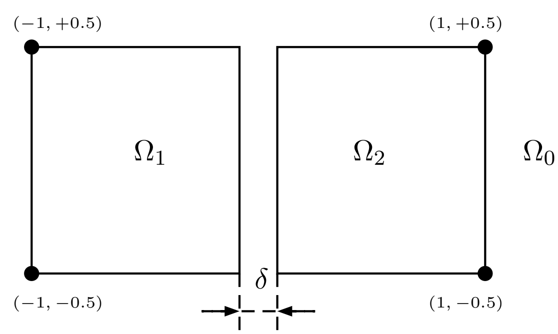
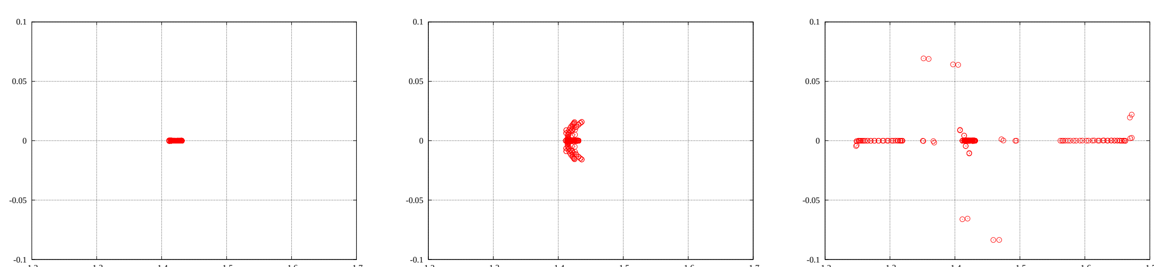


Figure 4: Spectrum of the local multi-trace operator in the case $\kappa_0 = 1$, $\kappa_1 = 5$ and $\kappa_2 = 2$ with $\alpha = 1$.

For the next series of figures, we consider the same scattering problem, but in a different geometrical configuration. The new configuration is depicted in the picture below: there are two square scatterer separated by a thin gap of width delta. In the theory we have presented, we needed to assume that there is no junction point i.e. points where at least three sub-domains abut. We wish to test the robustness, with respect to this assumption, of the theoretical formulas obtained.



In Figure 5, we consider $\kappa_0 = \kappa_1 = \kappa_2$ and $\alpha = 1$. For a fixed strictly positive value of $\delta > 0$, the eigenvalues are clustered around $\pm\sqrt{2}$. Each of the figures below represent a zoom at the cluster centered at $+\sqrt{2}$ for various values of δ.



(a) **(b)** **(c)**

Figure 5: Spectrum of the local multi-trace operator for $\kappa_0 = \kappa_1 = \kappa_2 = 1$, $\alpha = 1$ and three different values of gap: $\delta = 0.1$ (left) $\delta = 0.01$ (center) and $\delta = 0.001$ (right)

The cluster is more and more scattered as the gap closes, suggesting that the assumption that there is no junction point is mandatory, in spite of quadrature rules being less reliable as δ is close to 0. In the last picture below, we examine the case where the gap is closed i.e. $\delta = 0$, which corresponds to the presence of a junction point in the geometry.

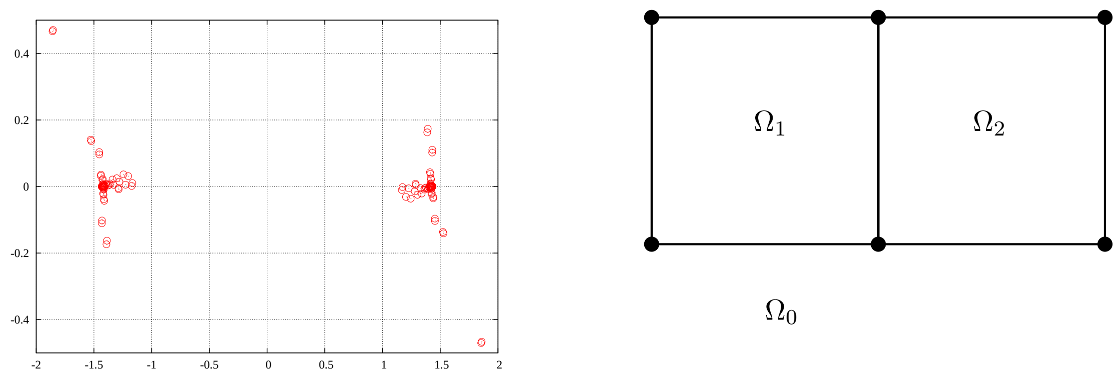


Figure 6: Spectrum of the local multi-trace operator (left) for $\kappa_0 = \kappa_1 = \kappa_2 = 1$, $\alpha = 1$ in the presence of junction points in the geometry (right).

Unfortunately, geometrical configurations involving junction points are not covered by the theory of the present article. However the result of Figure 6 suggests that, although the eigenvalues are not anymore closely clustered around the values $-1 + \alpha \pm \sqrt{1 + \alpha^2}$, these two theoretical points remain accumulation points of the spectrum of the local multi-trace operator.

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